Universality or Non–Universality of Absorption Cross Sections for Extended Objects

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Abstract

The calculation of absorption cross sections for minimal scalars in supergravity backgrounds is an important aspect of the investigation of AdS/CFT correspondence and requires a matching of appropriate wave functions. The low energy case has attracted particular attention. In the following the dependence of the cross section on the matching point is investigated. It is shown that the low energy limit is independent of the matching point and hence exhibits universality. In the high energy limit the independence is not maintained, but the result is believed to possess the correct energy dependence.

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After the entropy problem was solved within the framework of string theory [1] by identifying extremal black holes with BPS states, recent interest seems to be shifted to the Hawking radiation problem [2]. In this context the absorption cross section of extended objects has been computed in the framework of various models [3–9] requiring matching of wave functions, and the result always coincides with the area of the horizon up to a constant in the low energy limit. This universality[1] is examined in general for a spherically symmetric and asymptotically flat geometry [10], and is in addition generalized by computing the frequency–dependent leading order [11].

In this letter we argue that this universality property at low energy is related to the insensitivity of extended objects to the matching equations between the asymptotic solution ϕ_ω^∞ and the near-horizon solution ϕ_ω^{near}. Although this universal property disappears in the high enery limit, it will be shown that even in this case one can obtain the important information, *i.e.* the explicit energy-dependence of the absorption rate. Also, it is briefly shown that the universality property is maintained for the massive scalar case also.

We consider a massless scalar field Φ minimally coupled to a spherically symmetric geometry

$$ds^2 = \gamma_{\mu\nu}(r)dx^\mu dx^\nu + f(r)dr^2 + r^2 h(r) d\Omega_{n+1} \tag{1}$$

where $\gamma_{\mu\nu}(r)(\mu, \nu = 0, \cdots, p)$ is the metric on a $(p+1)$–dimensional world volume of the extended objects. The geometry is assumed to be asymptotically flat: $\gamma_{\mu\nu}(r) \to \eta_{\mu\nu}$, $f(r), g(r) \to 1$ as $r \to \infty$. Introducing a tortoise coordinate r^* defined by $dr^* \equiv dr\sqrt{-\gamma^{tt}(r)f(r)}$, and considering only s-waves, *i.e.* $\Phi = e^{-i\omega t}\phi_\omega(r)$, one can derive a differential equation similar to the Schrödinger equation

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[1]While the conventional meaning of universality indicates that the low energy cross section coincides with the area of the horizon, we will use this terminology when the low energy cross section exhibits a common behavior.

$$\left[-\frac{d^2}{dr^{*2}}+V\right]\psi=\omega^2\psi \tag{2}$$

where

$$\begin{aligned}
\phi_\omega &= \frac{1}{\sqrt{U}}\psi, \\
U &= \left[\gamma\gamma^{tt}\left[r^2h(r)\right]^{n+1}\right]^{\frac{1}{2}}, \\
V &= \frac{1}{\sqrt{U}}\frac{d^2\sqrt{U}}{dr^{*2}} \overset{r^*\to\infty}{\longrightarrow} \frac{n^2-1}{4r^2}.
\end{aligned} \tag{3}$$

Here, $\gamma \equiv det\gamma_{\mu\nu}$. The solution at $r\to\infty$ is easily obtained in terms of Bessel functions,

$$\phi_\omega^\infty = \frac{1}{(\omega r)^{n/2}}\left[AJ_{\frac{n}{2}}(\omega r)+BJ_{-\frac{n}{2}}(\omega r)\right] \tag{4}$$

for odd n. Of course, for even n the Bessel function with negative order has to be replaced by the Neumann function. Since the final forms of the absorption cross section are always equivalent, we will consider only the $n=odd$ case. Using the asymptotic formula of the Bessel function it is straightforward to derive incoming and outgoing fluxes;

$$\begin{aligned}
\mathcal{F}_\infty^{in} &= -\frac{1}{\pi\omega^n}\left[\mid A\mid^2+\mid B\mid^2+AB^*e^{i\frac{n}{2}\pi}+A^*Be^{-i\frac{n}{2}\pi}\right] \\
\mathcal{F}_\infty^{out} &= \frac{1}{\pi\omega^n}\left[\mid A\mid^2+\mid B\mid^2+A^*Be^{i\frac{n}{2}\pi}+AB^*e^{-i\frac{n}{2}\pi}\right].
\end{aligned} \tag{5}$$

In order to obtain the near–horizon solution we introduce several parameters as in Ref. [11]:

$$\begin{aligned}
\lim_{r\to 0} U(r) &\approx Sr^{a-b} \\
\lim_{r\to 0}\sqrt{-\gamma^{tt}(r)f(r)} &\approx \frac{T}{r^{b+1}}
\end{aligned} \tag{6}$$

and we confine ourselves to the case of $0<b\le a$ [11]. Then, it is straightforward to derive a near–horizon solution in terms of a Hankel function,

$$\phi_\omega^{near} \approx \frac{1}{(\omega r)^{\frac{a}{2}}}H_{\frac{a}{2b}}^{(2)}\left(\frac{\omega T}{br^b}\right) \tag{7}$$

and the incoming flux is

$$\mathcal{F}^{near} = \frac{4bS}{\pi\omega^a T}. \tag{8}$$

In deriving Eq.(7) we used the boundary condition that, as r approaches zero, the field contains only incoming waves. Since the absorption cross section per unit volume is defined as

$$\sigma = \frac{(2\pi)^{n+1}}{\omega^{n+1}\Omega_{n+1}} \mid \frac{\mathcal{F}^{near}}{\mathcal{F}^{in}_{\infty}} \mid \tag{9}$$

where $\Omega_{n+1} = 2\pi^{1+\frac{n}{2}}/\Gamma(1+\frac{n}{2})$, it is completely determined by determining the coefficients A and B through a matching equation. As claimed above we will show that the absorption cross section in the low energy limit is very insensitive to the choice of matching equation, which results in the universality of the low–energy absorption cross section.

To show this we assume there is a matching point in the finite region, say at $r = R$ and use the usual quantum mechanical matching method, *i.e.* continuity of the wave function and its derivative;

$$\begin{aligned} \phi^{\infty}_{\omega}(R) &= \phi^{near}_{\omega}(R) \\ \frac{d}{dR}\phi^{\infty}_{\omega}(R) &= \frac{d}{dR}\phi^{near}_{\omega}(R). \end{aligned} \tag{10}$$

Then it is straightforward to obtain A and B:

$$\begin{aligned} A &= (-1)^{\frac{n-1}{2}} \frac{\pi R(\omega R)^{\frac{n-a}{2}}}{2} \Bigg[\frac{n-a}{2R} J_{-\frac{n}{2}}(\omega R) H^{(2)}_{\frac{a}{2b}}\left(\frac{\omega T}{bR^b}\right) - \omega J'_{-\frac{n}{2}}(\omega R) H^{(2)}_{\frac{a}{2b}}\left(\frac{\omega T}{bR^b}\right) \\ &\qquad - \frac{\omega T}{R^{b+1}} J_{-\frac{n}{2}}(\omega R) H^{(2)\prime}_{\frac{a}{2b}}\left(\frac{\omega T}{bR^b}\right) \Bigg] \\ B &= (-1)^{\frac{n+1}{2}} \frac{\pi R(\omega R)^{\frac{n-a}{2}}}{2} \Bigg[\frac{n-a}{2R} J_{\frac{n}{2}}(\omega R) H^{(2)}_{\frac{a}{2b}}\left(\frac{\omega T}{bR^b}\right) - \omega J'_{\frac{n}{2}}(\omega R) H^{(2)}_{\frac{a}{2b}}\left(\frac{\omega T}{bR^b}\right) \\ &\qquad - \frac{\omega T}{R^{b+1}} J_{\frac{n}{2}}(\omega R) H^{(2)\prime}_{\frac{a}{2b}}\left(\frac{\omega T}{bR^b}\right) \Bigg] \end{aligned} \tag{11}$$

where a prime denotes differentiation with respect to the argument. Using Eq.(11) one can plot the cross section for different values of R as shown in Fig. 1. Fig. 1 demonstrates the important fact that the low energy cross section is independent of the choice of the matching point R, which is the origin of the idea of universality.

To show this more explicitly we compute the coefficients A and B in the low energy limit by using the asymptotic formulas of Bessel and Hankel functions:

$$
\begin{aligned}
A &= (-1)^{\frac{n-1}{2}} i n \omega^{-\frac{a}{2}} 2^{\frac{n}{2}-1} \left(\frac{2b}{\omega T}\right)^{\frac{a}{2b}} \frac{\Gamma\left(\frac{a}{2b}\right)}{\Gamma\left(1-\frac{n}{2}\right)} \\
B &= 0
\end{aligned}
\tag{12}
$$

at the leading order. One should note that the R-dependence disappears in A and B. Computing the low energy cross section using Eq.(12), one can obtain straightforwardly

$$
\sigma_L = \frac{\pi}{\Gamma^2\left(\frac{a}{2b}\right)} S\Omega_{n+1} \left(\frac{\omega T}{2b}\right)^{\frac{a}{b}-1}
\tag{13}
$$

which coincides with the result of Ref. [11]. Of course, when $a = b$, σ_L becomes $S\Omega_{n+1}$ which is an area of horizon.

One may argue that this R-dependence of σ_L is a special property of the matching equation (10). To disprove this one may choose other matching equations such as

$$
\begin{aligned}
&\left|\frac{\mathcal{F}_\infty^{out}}{\mathcal{F}_\infty^{in}}\right| + \left|\frac{\mathcal{F}^{near}}{\mathcal{F}_\infty^{in}}\right| = 1 \\
&\phi_\omega^\infty(R) = \phi_\omega^{near}(R).
\end{aligned}
\tag{14}
$$

However, after tedious calculation one can show that this matching equation also leads to Eq. (13) in the low energy limit. We think this insensitivity of extended objects to the choice of matching equations and matching points results in the universality of this limit.

However, the situation is completely different in the high energy limit. Taking the high energy limit,*i.e.* $\omega \to \infty$, in Eq. (11), one can obtain

$$
\begin{aligned}
A &= (-1)^{\frac{n-1}{2}} (\omega R)^{\frac{n-a+1}{2}} \left(\frac{bR^b}{\omega T}\right)^{\frac{1}{2}} e^{-i\left[\frac{\omega T}{bR^b} - \frac{\pi}{4}\left(1+\frac{a}{b}\right)\right]} \\
&\quad \times \left[\sin\left(\omega R + \frac{n-1}{4}\pi\right) + \frac{iT}{R^{b+1}} \cos\left(\omega R + \frac{n-1}{4}\pi\right)\right] \\
B &= (-1)^{\frac{n+1}{2}} (\omega R)^{\frac{n-a+1}{2}} \left(\frac{bR^b}{\omega T}\right)^{\frac{1}{2}} e^{-i\left[\frac{\omega T}{bR^b} - \frac{\pi}{4}\left(1+\frac{a}{b}\right)\right]} \\
&\quad \times \left[\sin\left(\omega R - \frac{n+1}{4}\pi\right) + \frac{iT}{R^{b+1}} \cos\left(\omega R - \frac{n+1}{4}\pi\right)\right],
\end{aligned}
\tag{15}
$$

which yields the absorption cross section σ_H in the high energy limit to be

$$\sigma_H = \frac{(2\pi/\omega)^{n+1}}{\Omega_{n+1} R^{n-a}} \frac{4S/T}{\left[\sqrt{\frac{R^{b+1}}{T}} - \sqrt{\frac{T}{R^{b+1}}}\right]^2}. \tag{16}$$

The appearance of R in Eq. (16) indicates the high energy cross section loses the universality property. However, the ω-dependence of σ_H, *i.e.* $\sigma_H \propto \omega^{-(n+1)}$, exhibits a decreasing behavior. This decreasing behavior in the high energy limit is also found numerically in Ref. [7]. This can be an important property as we learned from blackbody radiation. One may question the credibility of this ω-dependence in view of the R-dependence of σ_H. In fact, this is our belief, and the rigorous proof is still an open problem. However, one can achieve some more credibility by considering more complicated situations such as a fixed scalar whose low energy cross section does not exhibit a universality [12]. The authors of Ref. [12] compute the low energy absorption cross section by matching ϕ_ω^{near} and ϕ_ω^{∞} through the solution in the intermediate region as Unruh did in his seminal paper [13] and obtained $\sigma_s = 2\pi\omega^2$ for the s-wave. If one applies our matching method to this problem, $\sigma = 2\pi\omega^2 R^2/(R-1)^2$ is obtained. Although the explicit dependence on R in σ indicates the non-universality in this case, apart from this R-dependent factor the cross section exhibits the correct ω-dependence. This is the reason why we can have confidence in the ω-dependence of σ_H in Eq. (16).

Finally, we comment on the absorption cross section for the case of a massive scalar. It is interesting to know whether the universal property of the low energy cross section is still maintained or not. In this case the potential in Eq. (3) is changed to

$$V = \frac{1}{\sqrt{U}} \frac{d^2\sqrt{U}}{dr^{*2}} - \frac{m^2}{\gamma^{tt}} \tag{17}$$

where m is the mass of the scalar field. The asymptotic solution in this case is the same as that of Eq. (4) if ωr is replaced by $\omega v r$ where $v = \sqrt{1 - m^2/\omega^2}$.

If we assume $\lim_{r\to 0} \gamma^{tt}(r) \approx -W/r^{2c}$ where W and c are some constants, the potential of Eq. (2) in the $r \to 0$ region becomes of the form

$$V = V_1(r) + V_2(r) \tag{18}$$

where

$$
\begin{aligned}
V_1(r) &= \frac{a^2 - b^2}{4T^2} r^{2b} \\
V_2(r) &= \frac{m^2}{W} r^{2c}.
\end{aligned} \tag{19}
$$

We consider only the $b < c$ case for simplicity. The full description of the massive scalar case will be discussed elsewhere. Then we can take $V \approx V_1$ approximately, and hence the near-horizon solution is unchanged. This means the mass effect is decoupled in this case in the $r \approx 0$ region.

By applying our method it is straightforward to obtain the low energy cross section $\sigma_L^m = v^n \sigma_L$. Of course, σ_L^m becomes v^n times the area of the horizon when $a = b$. Also, in the high energy limit we can obtain the same cross section as that of Eq. (16) if R^{b+1} in the square root is replaced by vR^{b+1}.

In conclusion we make the following remarks. The explicit and exact calculation of S–matrices for specific potentials is generally only possible in some special cases and requires a detailed study of the solutions of the appropriate wave equation in adjoining domains of validity over the entire range of the variable. This old problem which in the past was studied in $1 + 3$ dimensions has received fresh impetus from the string theory interest in absorption cross sections and also for other and higher dimensions. In the special case of the $D3$ brane the absorption cross section can be calculated explicitly in terms of modified Mathieu functions in both the low and the high energy domains [8,9]. The matching of different branches of the solutions in domains of overlap can be done but is nontrivial. It is natural, therefore, particularly if one is interested, for instance, only in the low energy case, to devise simpler methods for the derivation. The method of using Bessel and Hankel functions is such a method, and has been employed particularly frequently in this context for asymptotically flat metrics [6,14]. However, since the matching point can be chosen arbitrarily, one wants to be ascertained that the result does not depend on its choice. In the above we demonstrated for a wide class of metrics the universality of the low energy result, i.e. its independence of the matching point. Applying the method to the high energy domain we see that there this universality does not ensue, although the energy dependence is expected to be correct. In a further extension, the method has also been applied above to massive scalars. Our findings are therefore particularly reensuring for the application of the simple Bessel function method to the low energy case. In view of the possible wide applicability of the method, the demonstration of universality is also of general interest.

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Figure Caption:

Fig. 1

The absorption cross section for $n = 1, b = T = s = 1$ and $a = 1$ and 2 for $R = 1, 2, 3, 4$.

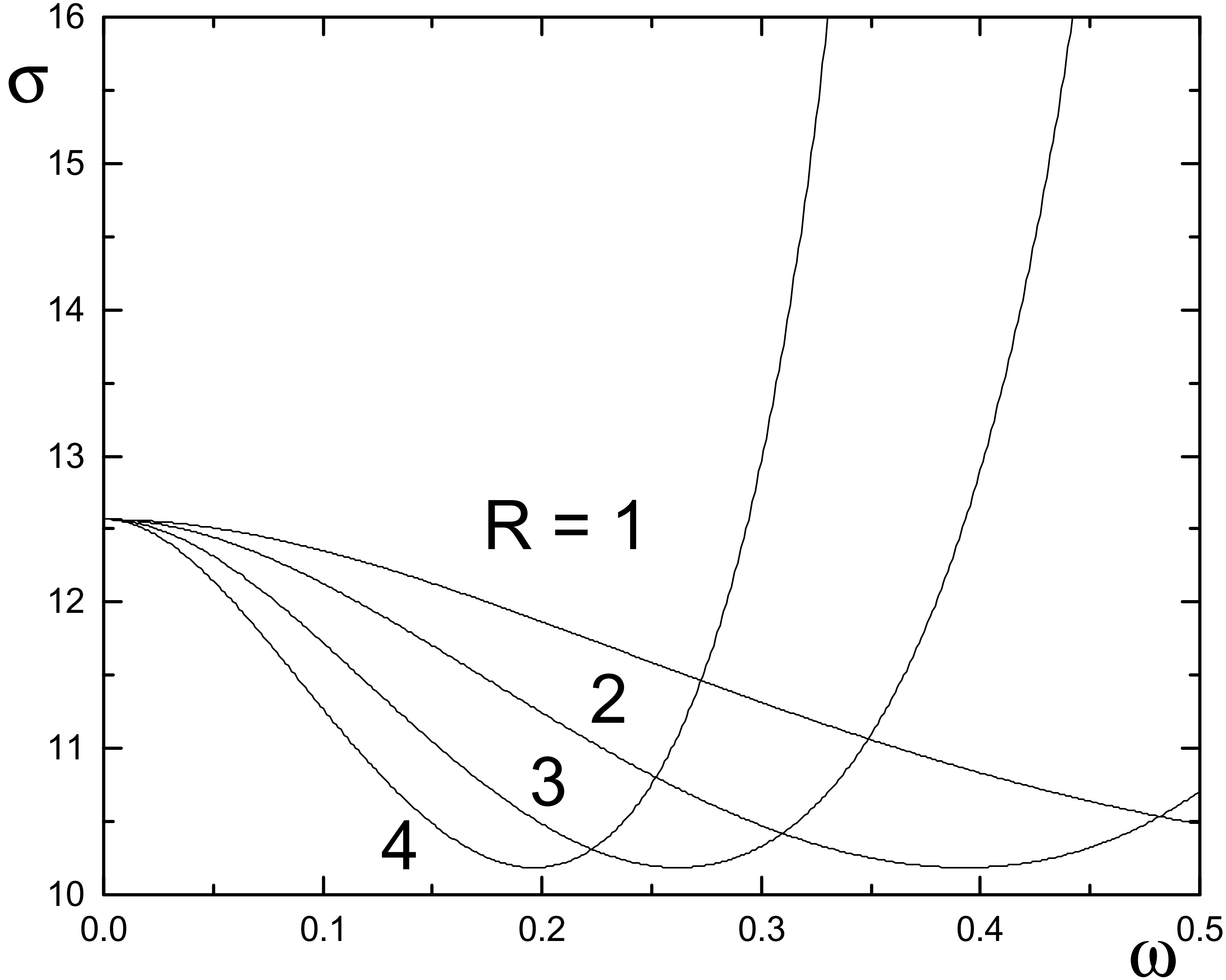

Fig. 1 (a)

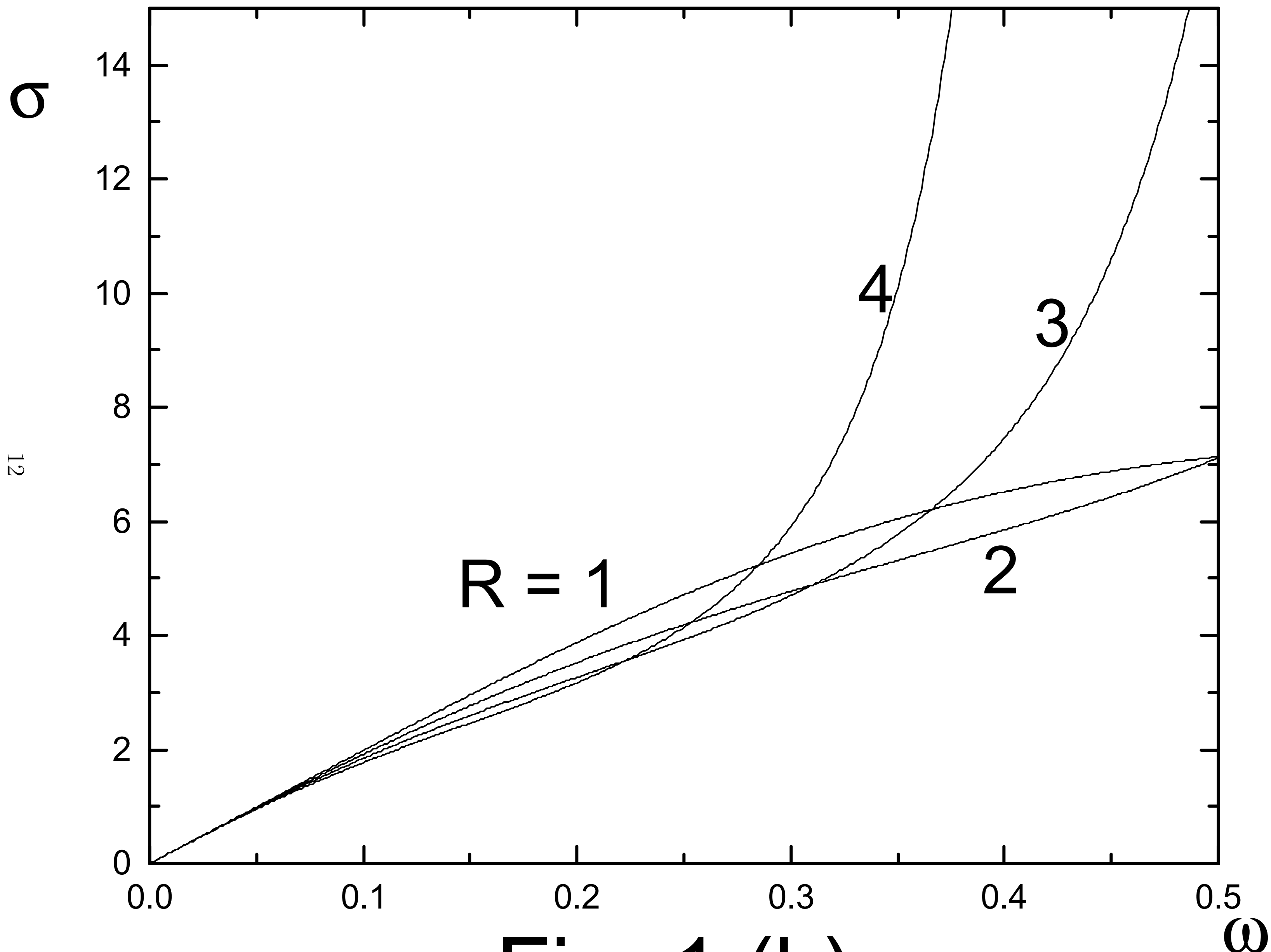

Fig. 1 (b)